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On the canonical variational 2-form and the canonical transformation of fields

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Abstract. Let field variables $\xi(x) = (q, p)$ be a set of coordinates on the ∞^{2n}-dimensional cotangent bundle T^*M and $\boldsymbol{V}_F$, $\boldsymbol{V}_G$ the Hamiltonian flow vectors of T^*M. The canonical variational 2-form $\tilde{\omega}^2(\xi)$ is defined by

$$\tilde{\omega}^2(\xi)(\boldsymbol{V}_F, \boldsymbol{V}_G) = \int \mathrm{d}^n x \left(\frac{\delta G}{\delta q^i} \frac{\delta F}{\delta p_i} - \frac{\delta G}{\delta p_i} \frac{\delta F}{\delta q^i} \right)$$

and obtained as $\tilde{\omega}^2(\xi) = \int \mathrm{d}^n x\, \tilde{\delta} p_i \wedge \tilde{\delta} q^i$. The condition of the field canonical transformation $g : \xi \mapsto \eta$ becomes $\tilde{\omega}^2(\xi) = \tilde{\omega}^2(\eta)$. The general theory of canonical transformations of fields is established. In particular, some examples of solving field equations and of field quantisation are given.

1. Introduction

The Hamiltonian expression of classical mechanics has two distinct advantages. First it is convenient for canonical quantisation. Second it allows us to make the canonical transformation. The use of exterior differential forms makes the advantages more obvious. The typical instances are the canonical transformation described by the canonical 2-form [1] and the achievement of geometric quantisation.

In field theory, the canonical transformation is ignored. The cause is probably that the canonical transformation of fields has some particular difficulties indicated by Goldstein in his classical mechanics textbook [2]. In this paper, we will show that those difficulties may be overcome and establish therefore the theory of canonical transformation of fields.

We give first the definitions of the basic variational form and the canonical variational 2-form. These are the direct extension of the corresponding differential forms. Then we prove the canonical transformation theorem, namely that the Poisson brackets of fields or canonical variational 2-form are invariant under the canonical transformation. We also discuss the Lagrange bracket of fields, the infinitesimal canonical transformation and the generating functional. Lastly, by using the theory of field canonical transformation, we easily solve a field equation and give the canonical transformation method of field quantisation.

In the paper, $x = (\boldsymbol{x}, -t) = (x^i, x^0)$ denote the $(n+1)$-dimensional spacetime coordinates. The field variables $q(x)$, $Q(x)$ and their conjugate canonical momenta $p(x)$, $P(x)$ are identified as

$$\begin{aligned} &\xi^\alpha(x) = q^i(x) \qquad &&\eta^\alpha(x) = Q^i(x) \qquad &&\alpha = m + i \\ &\xi^\alpha(x) = p_i(x) \qquad &&\eta^\alpha(x) = P_i(x) \qquad &&\alpha = i \end{aligned} \tag{1}$$

which are two sets of coordinates on the non-countable ∞^{2n}-dimensional cotangent bundle T^*M, where m is the number of field components.

Throughout the paper we adopt a summation convention for repeated indices: a greek index runs from 1 to $2m$ and any other index run from 1 to m unless it is particularly stated otherwise. The tilde above a letter denotes the variational form and bold face denotes the Hamiltonian flow vector [3].

2. Canonical variational 2-form

For brevity we introduce the matrix

$$\gamma = \begin{pmatrix} 0 & \mathrm{I} \\ -\mathrm{I} & 0 \end{pmatrix} = (\gamma_{\alpha\beta})$$

where by I we mean the $m \times m$ unit matrix. Using γ and (1) we can simply give the following definitions.

Definition 1. For any functional $F[\xi(x)]$ and the parameter s, the Hamiltonian flow vector is

$$\boldsymbol{V}_F = \frac{\partial \xi^\alpha}{\partial s} \boldsymbol{e}_\alpha = \gamma_{\beta\alpha} \frac{\delta F}{\delta \xi^\beta} \boldsymbol{e}_\alpha = V^\alpha_F \boldsymbol{e}_\alpha. \tag{2}$$

Definition 2. The basic variational form $\tilde{\delta}\xi^\alpha$ is a linear real function $\tilde{\delta}\xi^\alpha$: $T^*M \to R$ which satisfies the following:

(i) $\tilde{\delta}\xi^\alpha(\boldsymbol{V}_F) = V^\alpha_F$ where the $\tilde{\delta}\xi^\alpha$ are dual to $\boldsymbol{e}_\alpha$ (3)

(ii) all of the $\tilde{\delta}\xi^\alpha$ $(\alpha = 1, \ldots, 2m)$ and their exterior products obey the rules of the Grassmann algebra.

Definition 1 is the extension of the Hamiltonian flow vector in classical mechanics and definition 2 is the extension of the differential form [4]. From definition 2 we have

$$\tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta = -\tilde{\delta}\xi^\beta \wedge \tilde{\delta}\xi^\alpha \tag{4}$$

$$\tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta(\boldsymbol{V}_1, \boldsymbol{V}_2) = \det\begin{bmatrix} \tilde{\delta}\xi^\alpha(\boldsymbol{V}_1) & \tilde{\delta}\xi^\beta(\boldsymbol{V}_1) \\ \tilde{\delta}\xi^\alpha(\boldsymbol{V}_2) & \tilde{\delta}\xi^\beta(\boldsymbol{V}_2) \end{bmatrix} \tag{5}$$

and so on. Further extending the differential form, we obtain another definition.

Definition 3. The variational k-form in the ∞^{2n}-dimensional cotangent bundle T^*M is

$$\tilde{\omega}^k = \int \mathrm{d}^n x\, \omega_{\alpha_1 \ldots \alpha_k} \tilde{\delta}\xi^{\alpha_1} \wedge \ldots \wedge \tilde{\delta}\xi^{\alpha_k} \tag{6}$$

where $\int \mathrm{d}^n x\, \omega_{\alpha_1 \ldots \alpha_k}$ is a functional of ξ.

According to (6), the functional $F[\xi(x)]$ is a 0-form, the variational 1-form and 2-form can be written in general as

$$\tilde{\omega}^1 = \int \mathrm{d}^n x \quad \omega_\alpha \tilde{\delta}\xi^\alpha \qquad \tilde{\omega}^2 = \int \mathrm{d}^n x\, \omega_{\alpha\beta} \tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta. \tag{7}$$

In order to evade the question of defining the functional product, we avoid the definition of exterior multiplication. Here, we give the definition of the variation of a form.

Definition 4. The variation of the k-form is a $(k+1)$-form, i.e.

$$\tilde{\delta}\tilde{\omega}^k = \int \mathrm{d}^n x \frac{\delta \omega_{\alpha_1 \ldots \alpha_k}}{\delta \xi^\beta} \tilde{\delta}\xi^\beta \wedge \tilde{\delta}\xi^{\alpha_1} \wedge \ldots \wedge \tilde{\delta}\xi^{\alpha_k}. \tag{8}$$

Let $G[\xi(x)]$ be an arbitrary functional in T^*M; then application of (2) and (3) yields

$$\tilde{\delta}G(\boldsymbol{V}_F) = \int \mathrm{d}^n x \frac{\delta G}{\delta \xi^\alpha} \tilde{\delta}\xi^\alpha(\boldsymbol{V}_F) = \int \mathrm{d}^n x \frac{\delta G}{\delta \xi^\alpha} \gamma_{\beta\alpha} \frac{\delta F}{\delta \xi^\beta} = [G, F]_\xi \tag{9}$$

where the square bracket denotes the Poisson bracket. Making use of the properties of the Poisson bracket, we have manifestly the skew symmetric equation

$$\tilde{\delta}G(\boldsymbol{V}_F) = -\tilde{\delta}F(\boldsymbol{V}_G). \tag{10}$$

A special example of the functional is

$$F[\xi(x)] = \xi^\mu(x) = \int \mathrm{d}^n x' \, \delta^n(x'-x)\xi^\mu(x') \tag{11}$$

with Dirac's δ function $\delta^n(x'-x)$. Inserting this into (2) yields

$$\boldsymbol{V}_{\xi^\mu} = \gamma_{\beta\alpha} \frac{\delta \xi^\mu}{\delta \xi^\beta} \boldsymbol{e}_\alpha = \gamma_{\mu\alpha}\delta^n(x-x')\boldsymbol{e}_\alpha. \tag{12}$$

Let $\Delta\tau_\beta$ be the volume element of space, we can easily prove that $\boldsymbol{V}_{\xi^\mu}$ is a unit flow vector [5]

$$\boldsymbol{V}_{\xi^\mu} = \lim_{\Delta\tau_\beta \to 0} \frac{1}{\Delta\tau_\beta} \delta_{\beta\mu}\gamma_{\mu\alpha}\boldsymbol{e}_\alpha.$$

Given (12), we have the basic Poisson bracket

$$\tilde{\delta}\xi^\nu(\boldsymbol{V}_{\xi^\mu}) = [\xi^\nu(x), \xi^\mu(x')]_\xi = \gamma_{\mu\nu}\delta^n(x-x'). \tag{13}$$

Now let us give the definition of the canonical variational 2-form.

Definition 5. Let F and G be arbitrary functionals in T^*M, and let $\boldsymbol{V}_F$ and $\boldsymbol{V}_G$ be the corresponding flow vectors, then the canonical variational 2-form $\tilde{\gamma}(\xi)$ is defined by

$$\tilde{\gamma}(\xi)(\boldsymbol{V}_F, \boldsymbol{V}_G) = [G, F]_\xi. \tag{14}$$

Let us find the $\tilde{\gamma}(\xi)$. From the general form of $\tilde{\omega}^2$ in (7) and applying (3), (5) to (14), we obtain

$$\begin{aligned}
\tilde{\gamma}(\boldsymbol{V}_F, \boldsymbol{V}_G) &= \int \mathrm{d}^n x \, \omega_{\alpha\beta}\tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta(\boldsymbol{V}_F, \boldsymbol{V}_G) \\
&= \int \mathrm{d}^n x \, \omega_{\alpha\beta}[\tilde{\delta}\xi^\alpha(\boldsymbol{V}_F)\tilde{\delta}\xi^\beta(\boldsymbol{V}_G) - \tilde{\delta}\xi^\alpha(\boldsymbol{V}_G)\tilde{\delta}\xi^\beta(\boldsymbol{V}_F)] \\
&= \int \mathrm{d}^n x \, \omega_{\alpha\beta}\gamma_{\mu\alpha}\gamma_{\nu\beta}\left(\frac{\delta F}{\delta \xi^\mu}\frac{\delta G}{\delta \xi^\nu} - \frac{\delta F}{\delta \xi^\nu}\frac{\delta G}{\delta \xi^\mu}\right).
\end{aligned}$$

The right-hand side of (14) is given by (9). By comparing these we see that $\omega_{\alpha\beta}\gamma_{\mu\alpha}\gamma_{\nu\beta} = \frac{1}{2}\gamma_{\mu\nu}$. Since the matrix γ is orthogonal and antisymmetric, we therefore have $\omega_{\alpha\beta} = \frac{1}{2}\gamma_{\alpha\beta}$. Inserting this into (7), we obtain the canonical variational 2-form

$$\tilde{\gamma}(\xi) = \int \mathrm{d}^n x \, \tfrac{1}{2}\gamma_{\alpha\beta}\tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta. \tag{15}$$

It is obvious that the geometric sense of $\tilde{\gamma}$ is a sum of unit (q, p) planes.

3. Canonical transformation theorem of fields

Now we consider a coordinate transformation $\xi \to \eta$ in the cotangent bundle T^*M. For arbitrary functionals F and G of η, we define the Poisson bracket by

$$[F, G]_\eta = \int \mathrm{d}^n x \frac{\delta F}{\delta \eta^\mu} \gamma_{\nu\mu} \frac{\delta G}{\delta \eta^\nu}. \tag{16}$$

Then we have the canonical transformation theorem of fields.

Theorem. Let ξ be a set of canonical variables on T^*M, for arbitrary functionals F, G and non-zero constant Z; then the mapping $g : \xi \mapsto \eta$ is canonical if and only if

$$[F, G]_\eta = Z[F, G]_\xi. \tag{17}$$

This theorem can be proved by using the method which is similar to one used to prove a corresponding theorem in mechanics [6], because the Poisson brackets of field theory and mechanics have the same algebraic properties.

In the η notation, the canonical variational 2-form is

$$\tilde{\gamma}(\eta) = \int \mathrm{d}^n x \, \tfrac{1}{2} \gamma_{\mu\nu} \tilde{\delta}\eta^\mu \wedge \tilde{\delta}\eta^\nu \tag{18}$$

the Hamiltonian flow vector generated by F is

$$\boldsymbol{V}_F = \frac{\partial \eta^\mu}{\partial s} \boldsymbol{e}_\mu = \gamma_{\nu\mu} \frac{\delta F}{\delta \eta^\nu} \boldsymbol{e}_\mu \tag{19}$$

and corresponding to (14) we have

$$\tilde{\gamma}(\eta)(\boldsymbol{V}_F, \boldsymbol{V}_G) = [G, F]_\eta. \tag{20}$$

Therefore from the theorem (17) we obtain a condition of canonical transformation in the form

$$\tilde{\gamma}(\eta) = Z \tilde{\gamma}(\xi) \tag{21}$$

i.e. the sum of unit (Q, P) planes is equal to the sum of unit (q, p) planes multiplied by Z.

Letting (21) act on the double unit vector $(\boldsymbol{V}_{\eta^\alpha}, \boldsymbol{V}_{\xi^\beta})$, we obtain the direct condition of canonical transformation [2]

$$\gamma_{\beta\mu} \frac{\delta \eta^\alpha}{\delta \xi^\mu} = Z \gamma_{\nu\alpha} \frac{\delta \xi^\beta}{\delta \eta^\nu}. \tag{22}$$

Expanding the left-hand side of (21) directly yields

$$\tilde{\gamma}(\eta) = \tfrac{1}{2} \int \mathrm{d}^n x \, \gamma_{\mu\nu} \tilde{\delta}\eta^\mu \wedge \tilde{\delta}\eta^\nu = \tfrac{1}{2} \int \mathrm{d}^n x \, \mathrm{d}^n x' \, \mathrm{d}^n x'' \, \gamma_{\mu\nu} \frac{\delta \eta^\mu}{\delta \xi^\alpha} \frac{\delta \eta^\nu}{\delta \xi^\beta} \tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta.$$

Comparing this with the right-hand side of (21), we obtain the basic Lagrange bracket of fields

$$\{\xi^\alpha(x), \xi^\beta(x')\} = \int \mathrm{d}^n x \, \gamma_{\mu\nu} \frac{\delta \eta^\nu}{\delta \xi^\alpha} \frac{\delta \eta^\mu}{\delta \xi^\beta} = Z \gamma_{\beta\alpha} \delta^n(x - x'). \tag{23}$$

Now we show that the infinitesimal canonical transformation satisfies (21). In the following discussion, the $k \geqslant 2$ order infinitesimals $(\Delta t)^k$ are omitted. We assume an infinitesimal transformation generated by Hamiltonian H in the form

$$\xi^\alpha = \xi_o^\alpha + \Delta t \dot{\xi}_o^\alpha = \xi_o^\alpha + \Delta t \gamma_{\nu\alpha} \frac{\delta H}{\delta \xi_o^\nu}.$$

Calculating the corresponding forms by $\tilde{\delta} F = \int \mathrm{d}^n x (\delta F / \delta \xi^\alpha) \tilde{\delta} \xi^\alpha$, we find

$$\tilde{\delta}\xi^\alpha = \tilde{\delta}\xi_o^\alpha + \Delta t \int \mathrm{d}^n x \, \gamma_{\nu\alpha} \frac{\delta^2 H}{\delta \xi_o^\mu \delta \xi_o^\nu} \tilde{\delta}\xi^\mu$$

where index o denotes the old coordinates. Using exterior products to form $\tilde{\gamma}$,

$$\begin{aligned}
\gamma_{\alpha\beta}\tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta &= \gamma_{\alpha\beta}\left(\tilde{\delta}\xi_o^\alpha + \Delta t \int \mathrm{d}^n x' \, \gamma_{\nu\alpha} \frac{\delta^2 H}{\delta \xi_o^\mu \delta \xi_o^\nu} \tilde{\delta}\xi_o^\mu\right) \wedge \left(\tilde{\delta}\xi_o^\beta + \Delta t \int \mathrm{d}^n x'' \, \gamma_{\rho\beta} \frac{\delta^2 H}{\delta \xi_o^\rho \delta \xi_o^\sigma} \tilde{\delta}\xi_o^\sigma\right) \\
&= \gamma_{\alpha\beta}\tilde{\delta}\xi_o^\alpha \wedge \tilde{\delta}\xi_o^\beta + \gamma_{\alpha\beta}\Delta t \int \mathrm{d}^n x' \, \mathrm{d}^n x'' \\
&\quad \times \left(\gamma_{\nu\alpha} \frac{\delta^2 H}{\delta \xi_o^\mu \delta \xi_o^\nu} \tilde{\delta}\xi_o^\mu \wedge \tilde{\delta}\xi_o^\beta - \gamma_{\rho\beta} \frac{\delta^2 H}{\delta \xi_o^\rho \delta \xi_o^\sigma} \tilde{\delta}\xi_o^\alpha \wedge \tilde{\delta}\xi_o^\sigma\right) \\
&= \gamma_{\alpha\beta}\tilde{\delta}\xi_o^\alpha \wedge \tilde{\delta}\xi_o^\beta - 2\Delta t \int \mathrm{d}^n x' \, \mathrm{d}^n x'' \frac{\delta^2 H}{\delta \xi_o^\beta \delta \xi_o^\mu} \tilde{\delta}\xi_o^\beta \wedge \tilde{\delta}\xi_o^\mu .
\end{aligned}$$

The second term is zero because the functional derivative factor is symmetric under index interchange and the exterior products are antisymmetric. Therefore

$$\tilde{\gamma}(t) = \int \mathrm{d}^n x \, \gamma_{\alpha\beta}\tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta = \int \mathrm{d}^n x \, \gamma_{\alpha\beta}\tilde{\delta}\xi_o^\alpha \wedge \tilde{\delta}\xi_o^\beta = \tilde{\gamma}(0). \tag{24}$$

From this we see that the canonical variational 2-form is an invariant form in the motion. If we divide the T^*M into many cells formed by $(\delta\xi^i e_i, \delta\xi^{m+i} e_{m+i})$, then (24) implies that

$$\int \mathrm{d}^n x \, \gamma_{\alpha\beta}\tilde{\delta}\xi^\alpha \wedge \tilde{\delta}\xi^\beta = \text{constant}.$$

Let us show that the field canonical transformation may be generated by some generating functionals, as is the case for classical mechanics. Setting some 1-forms as the following:

$$\begin{aligned}
\tilde{\psi}_1 &= \xi^i \tilde{\delta}\xi^{m+i} + \tilde{\delta} f_1 & \qquad \tilde{\psi}_2 &= \xi^{m+i} \tilde{\delta}\xi^i + \tilde{\delta} f_2 \\
\tilde{\psi}_3 &= \eta^i \tilde{\delta}\eta^{m+i} + \tilde{\delta} f_3 & \qquad \tilde{\psi}_4 &= \eta^{m+i} \tilde{\delta}\eta^i + \tilde{\delta} f_4
\end{aligned} \tag{25}$$

since $\tilde{\delta}\tilde{\psi}_1 = -\tilde{\delta}\tilde{\psi}_2 = \tilde{\delta}\tilde{\psi}_3 = -\tilde{\delta}\tilde{\psi}_4$, we may obtain

$$\tilde{\psi}_1 = -\tilde{\psi}_2 = \tilde{\psi}_3 = -\tilde{\psi}_4. \tag{26}$$

We define the generating functional G_i as

$$G_1 = f_3 - f_1 \qquad G_2 = f_3 + f_2 \qquad G_3 = f_4 + f_1 \qquad G_4 = f_4 - f_2. \tag{27}$$

Applying (25), (26) and (27) we have the condition of canonical transformation in the new form

$$\begin{aligned} \tilde{\delta}G_1 &= \xi^i\tilde{\delta}\xi^{m+i} - \eta^i\tilde{\delta}\eta^{m+i} & \tilde{\delta}G_2 &= -\xi^{m+i}\tilde{\delta}\xi^i - \eta^i\tilde{\delta}\eta^{m+i} \\ \tilde{\delta}G_3 &= -\xi^i\tilde{\delta}\xi^{m+i} - \eta^{m+i}\tilde{\delta}\eta^i & \tilde{\delta}G_4 &= \xi^{m+i}\tilde{\delta}\xi^i - \eta^{m+i}\tilde{\delta}\eta^i. \end{aligned} \tag{28}$$

Using (21) with $Z = 1$ we find $\tilde{\delta}^2 G_i = 0$.

We see that the three conditions (22), (28) and (21) on the canonical transformation are respectively expressed by 0-form, 1-form and 2-form.

As is the case in the differential form, we have the Stokes theorem in variational form

$$\int_{\partial U} \tilde{\omega}^k = \int_U \tilde{\delta}\tilde{\omega}^k \tag{29}$$

where ∂U is the bound of range U in T^*M.

4. Canonical transformation approach to classical and quantum fields

In this section we study the applications of the method of field canonical transformation. We give two examples of classical and quantum fields. Through these examples, we point out the technique and meaning of applying the method of canonical transformation to fields.

4.1. Solving non-linear field equations with the canonical transformation

Considering the SG equation

$$\varphi_{tt} - \varphi_{xx} - \varphi_{yy} - \varphi_{zz} = \sin\varphi \tag{30}$$

in four-dimensional spacetime, we will find its plane soliton. The canonical field variables are

$$Q = \varphi \qquad P = \pi = \varphi_t.$$

We make a canonical transformation $g: (Q, P) \mapsto (q, p)$ such that

$$q = Ax + By + Cz + Dt \tag{31}$$

$$\begin{aligned} p &= \int \mathrm{d}^3x \frac{D}{k^2}\left(\tfrac{1}{2}\pi^2 - \tfrac{1}{2}\nabla\varphi\cdot\nabla\varphi + \cos\varphi\right) \\ &= \int \mathrm{d}^3x \left(\frac{1}{2D}\pi^2 + \frac{D}{k^2}\cos\varphi\right) \qquad k^2 = D^2 - A^2 - B^2 - C^2 \end{aligned} \tag{32}$$

where $\pi = D\,\mathrm{d}\varphi/\mathrm{d}q$. The density of the new generalised momentum p is

$$\not{p} = \frac{1}{2D}\pi^2 + \frac{D}{k^2}\cos\varphi. \tag{33}$$

We easily see that the transformation given above satisfies the direct condition (22) of canonical transformation, namely

$$\begin{aligned} \frac{\mathrm{d}\varphi}{\mathrm{d}q} &= \frac{\delta p}{\delta\pi} = \frac{\partial\not{p}}{\partial\pi} = \frac{\pi}{D} \\ \frac{\mathrm{d}\pi}{\mathrm{d}q} &= -\frac{\delta p}{\delta\varphi} = -\frac{\partial\not{p}}{\partial\varphi} + \partial_k\frac{\partial\not{p}}{\partial(\partial_k\varphi)} = -\frac{\partial\not{p}}{\partial\varphi} = \frac{D}{k^2}\sin\varphi \end{aligned} \tag{34}$$

because from (34) we can obtain $k^2(\mathrm{d}^2\varphi/\mathrm{d}q^2) = \sin\varphi$. This is the same as the result obtained by inserting (31) into (30). From (34) we find easily that

$$\frac{\mathrm{d}\wp}{\mathrm{d}t} = D\frac{\mathrm{d}\wp}{\mathrm{d}q} = D\left(\frac{\partial\wp}{\partial\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}q} + \frac{\partial\wp}{\partial\pi}\frac{\mathrm{d}\pi}{\mathrm{d}q}\right) = 0$$

this implies p and $\wp$ are the constants of motion. Then we can solve (33) for π and use it in (34), arriving at the plane-wave solution of (30):

$$q - q_0 = \pm\int\frac{\mathrm{d}\varphi}{\sqrt{(2/D)\wp - (2/k^2)\cos\varphi}} \tag{35}$$

By appropriately selecting D and k, from (35) we can easily obtain the plane soliton of the SG equation.

Applying the above-mentioned method, obviously, we can obtain the plane solitons of the non-linear KG equations obtained in many articles such as [7, 8] and the corresponding results for any n-dimensional case. Inspecting the example given above, we find that it has a few characteristics such as the following.

(a) The new canonical variables are without clear and definite physical meaning; this is in agreement with the corresponding results in classical mechanics. On one hand, the q in (31) is a generalised field coordinate; on the other hand, the space coordinates x also act as indices. Goldstein thinks that this cannot be accepted by the theory [2]. But, just as we see from (34), the relations between q and (φ, π) satisfy the condition of canonical transformation indeed. We must accept this fact! It looks as if we ought not to investigate the physical meaning of the new canonical variables.

(b) The new generalised momentum and its density are constants of the motion; this is one of the cruxes of simplifying the problems. The canonical transformations in mechanics are also done in this way. Seeking some constants of motion and taking them as the canonical variables can simplify the problem of solving field equations. The canonical transformation can otherwise make the problem more complicated.

(c) Evading the generating functional and making use of the condition of direct canonical transformation are another key to simplifying the problems. We have studied the similar case in mechanics [9].

According to the above-mentioned characteristics, we can rapidly find the appropriate new canonical variables and simply obtain the solutions for some field equation.

4.2. *The canonical transformation method of field quantisation*

In the canonical quantisation theory of classical dynamics, the rectangular coordinates x^i and their conjugate momenta p_i are the basic canonical variables. The Poisson brackets taking x^i, p_i as arguments are the basic brackets. For the basic Poisson brackets of the dynamic variables F and G, the canonical quantisation conditions are such that [10]

$$[F, G] = FG - GF = \mathrm{i}\hbar[F, G]_{sp} \tag{36}$$

where F, G are the corresponding Hermite operators.

As a dynamic system, the field has its energy-momentum [11] p_μ $(\mu = 0, 1, 2, 3)$. What is the conjugate canonical coordinate of p_i? According to the correspondence between field and dynamics, we may assert categorically that one conjugate to the momentum p_i of the field must be the space coordinate x^i! In fact, [11] and many other articles have used this assertion to handle the problem of the translation invariance of fields [12]. Maintaining (x, p) as the canonical variables of fields, from (22) we obtain the momentum equation

$$\frac{\delta p_i}{\delta \varphi^j} = -\frac{\partial \pi_j}{\partial x^i} \qquad \frac{\delta p_i}{\delta \pi_j} = \frac{\partial \varphi^j}{\delta x^i} \qquad i = 1, 2, 3 \quad j = 1, \ldots, m. \tag{37}$$

Conversely, from the course in field theory obtaining that (x, p) satisfy the direct condition (37) of the canonical transformation, we may assert that (x, p) are the canonical variables of fields. As for the sense of the generalised coordinates x^i of fields, we do not investigate this. As in the dynamic case, we maintain that (x, p) are the basic canonical variables of fields, then the quantisation conditions of the basic Poisson brackets of fields are just (36). Given (17) and (13), we easily obtain from (36) the commutation relation of the field variables (φ, π):

$$\begin{aligned}[\varphi^i(x), \pi_j(x')] &= \mathrm{i}\hbar[\varphi^i(x), \pi_j(x')]_{xp} = \mathrm{i}Z\hbar[\varphi^i(x), \pi_j(x')]_{\varphi\pi} \\ &= \mathrm{i}\hbar\delta_{ij}\delta^3(x - x') \qquad \text{for } Z = 1.\end{aligned} \tag{38}$$

Here, we have considered a restricted canonical transformation with $Z = 1$. This kind of restricted case is universal in classical dynamics.

If we now make a canonical transformation with $Z = \mathrm{i}$, then we have the annihilation and creation operators of the particles after the quantisation. This method of field quantisation is new and simple. A clear example on the canonical transformation with $Z = \mathrm{i}$ is the complex transformation [13].

As an example, let us consider a complex canonical transformation in matrix form [14]:

$$\eta^\alpha(k) = (2\pi)^{-3/2} \int \mathrm{d}^3x\, J_{\alpha\beta}(kx)\xi^\beta(x) \tag{39}$$

where $\xi = (\varphi, \pi)$ are the scalar field variables and k denotes the momentum of the particles. We require the energy-momentum of the fields in terms of the particle number operator, which retain the usual form

$$p_\mu = \int \mathrm{d}^3k\, k_\mu(N + \tfrac{1}{2}) = \int \mathrm{d}^3k\, \tfrac{1}{2}k_\mu(\eta^\alpha\eta^\beta + \eta^\beta\eta^\alpha)|\gamma_{\alpha\beta}| \qquad \mu = 0, 1, \ldots, n. \tag{40}$$

Thus $J_{\alpha\beta}(kx)$ in (39) must satisfy the following relations:

$$\begin{aligned} &J_{\alpha i} = -J^*_{\alpha i} \qquad J_{\alpha, m+i} = J^*_{\alpha, m+i} \\ &J_{ji} = -J_{m+j, i} \qquad J_{j, m+i} = J_{m+j, m+i} \\ &|\gamma_{\alpha\beta}|J_{\alpha\mu}(kx)J_{\beta\nu}(k'x) = \delta_{\mu\nu}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{x}} \\ &\gamma_{\mu\nu}J_{\alpha\nu}(kx)J_{\beta\mu}(k'x) = \mathrm{i}\gamma_{\beta\alpha}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{x}} \\ &j, k = 1, \ldots, m \qquad \alpha, \beta, \mu, \nu = 1, \ldots, 2m. \end{aligned} \tag{41}$$

Therefore we have the Poisson brackets

$$\begin{aligned}[\eta^\alpha(k), \eta^\beta(k')]_\xi &= (2\pi)^{-3} \int \mathrm{d}^3x\, \gamma_{\mu\nu}J_{\alpha\nu}(kx)J_{\beta\mu}(k'x) \\ &= \int \mathrm{d}^3x(2\pi)^{-3}\mathrm{i}\gamma_{\beta\alpha}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{x}} = \mathrm{i}\gamma_{\beta\alpha}\delta^3(\boldsymbol{k} - \boldsymbol{k}').\end{aligned} \tag{42}$$

After the quantisation this becomes

$$[\eta^{\alpha}(k), \eta^{\beta}(k')] = \gamma_{\alpha\beta}\delta^{3}(\boldsymbol{k}-\boldsymbol{k}') \tag{43}$$

where the η are just some kind of creation and annihilation operators. The difference between η and the corresponding operators [11] (a, a^{*}) in the general field theory lies in the fact that any η^{α} is the linear combination of all ξ and any a_i is only related to part of ξ.

The inverse transformation of (39) is

$$\xi^{\alpha}(x) = (2\pi)^{-3/2} \int \mathrm{d}^{3}k\, J^{-1}_{\alpha\beta}(kx)\eta^{\beta}(k). \tag{44}$$

This shows that any φ^{i} relates to all of the creation and annihilation operators η, but the φ^{i} in (a, a^{*}) notation does not have this property. A matter deserving of attention is the fact that, in the η notation, although the form of energy-momentum of the free fields in term of the particle number operator $N = \eta^{i}\eta^{m+i} = a_i^{*}a_i$ is not changed, the form of the Hamiltonian of the interactive fields will be changed. This will directly affect the calculating of results for the scattering matrix elements. We wish by applying the method to improve the original results and by making the linear or non-linear canonical transformation to eliminate some divergent terms in the scattering matrix elements.

In addition, applying the theory established above to the calculus of variations and the variational principle of fields [15], there is still more interesting work. From [16] we may see the gauge transformation in field theory as a canonical transformation. This will be studied further in future papers.

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